WAVE STRUCTURE OF TURBULENT MIXING ZONE

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Under consideration is a method based on the quantum analogy of turbulence and its application to the problem of a turbulent mixing layer.

The quantum analogy of motion can inherently be applied to turbulence as a wave phenomenon with a probabilisitic nature of fluctuations, inasmuch as vortices in a large array behave like wave–particle formations [1]. A change in the state of a vortex, that is a fluid "quantum" particle of unit volume, is defined by the equation

$$ih\frac{\partial\psi}{\partial t} = -\frac{h^2}{2\rho}\nabla^2\psi, \quad i = \sqrt{-1}. \tag{1}$$

Here and henceforth ∇^2 denotes the Laplace operator. We now represent the wave function in the form $\psi = a\exp(ib)$ so that, after separating its real and imaginary parts in Eq. (1), we obtain

$$-h\frac{\partial b}{\partial t} = \frac{p^2}{2\rho} - \frac{h^2}{2\rho}\frac{\nabla^2 a}{a}, \tag{2}$$

$$h\,\mathrm{div}\,(a^2\,\mathrm{grad}\,b) = \rho\frac{\partial a^2}{\partial t}, \tag{3}$$

where $p = h|\mathrm{grad}\ b|$, and grad b is the wave vector. Passage to the limit on the side of nonturbulent flow ($h \to 0$, $b \to \infty$) reveals that Eq. (2) will express the kinetic energy of a vortex $E = -h\partial b/\partial t$ ($-\partial b/\partial t$ denoting the fluctuation frequency) in corpuscular-wave terms if $p = \rho U$. When $h \neq 0$, then

$$E = \frac{p^2}{2\rho} + \Delta E, \tag{4}$$

with the quantity $\Delta E = -(h^2/2\rho)a^{-1}\nabla^2 a$ representing the specific turbulent energy increment and also, because of the statistical nature of irregular vortex flow through a background of average motion, representable as the mathematical expectation of the random energy spread. Postulating that the quantity a^2 represents the probability density of the excited state of a vortex (i.e., characterizes its proneness to oscillatory excitation), one can write $\Delta E = a^2p^2/2\rho$. Combining these two definitions of ΔE yields

$$\nabla^2 a + |\mathrm{grad}\,b|^2 a^3 = 0. \tag{5}$$

Since excitation of a vortex is likely to occur (a vortex is necessarily in some phase of oscillation), we have

$$\int a^2 db = 1, \tag{6}$$

with the integral taken over all phases within the region of vortex existence. The probability flux a^2 is described by Eq. (3). In the case of steady turbulence $\partial a^2/\partial t = 0$.

Along one axis of coordinates s the quantity $k = |\partial b/\partial s|$ is the wave number and the wave front describable by the ψ-function can be represented as

$$\psi(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)\exp(iks)\,dk, \quad F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\psi(s)\exp(-iks)\,ds, \tag{7}$$

with $\psi(s) \to 0$ as $s \to \infty$ and $s \to -\infty$ (the wave motion vanishes beyond the turbulence zone, where $a = 0$). Normalization (6) implies the existence of correspondence relations

$$\int_{-\infty}^{\infty}\psi(s)\,\psi^*(s)\,ds = N, \quad \int_{-\infty}^{\infty} F(k)\,F^*(k)\,dk = N, \tag{8}$$

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where N is the normalizing multiplier and $\psi^*(s)$, $F^*(k)$ are complex-conjugate functions. Considering that s is measured from the center of the wave packet, according to relations (8), one can write for the mean-square displacement of a vortex $\langle s^2 \rangle$ and for the corresponding spread of the wave number squared $\langle k^2 \rangle$

$$N \langle s^2 \rangle = \int_{-\infty}^{\infty} s^2 \psi(s) \psi^*(s)\, ds, \tag{9}$$

$$N \langle k^2 \rangle = \int_{-\infty}^{\infty} k^2 F(k) F^*(k)\, dk, \tag{10}$$

respectively. With the representation (7), the integral in expression (10) transforms into

$$N \langle k^2 \rangle = -\int_{-\infty}^{\infty} \psi^*(s) \frac{\partial^2 \psi}{\partial s^2}\, ds. \tag{11}$$

There exists a constraint of the indeterminacy kind [2], namely

$$\langle s^2 \rangle \langle k^2 \rangle \geqslant \frac{1}{4}. \tag{12}$$

This can be verified by considering the G. Weil expression

$$I(\varepsilon) = \int_{-\infty}^{\infty} \left(s\psi + \varepsilon \frac{\partial \psi}{\partial s}\right)\left(s\psi^* + \varepsilon \frac{\partial \psi^*}{ds}\right) ds, \tag{13}$$

which is positive-definite for any ε. Considering (9) and (11), we obtain $I(\varepsilon) = N(\langle s^2 \rangle - \varepsilon + \langle k^2 \rangle \varepsilon^2)$ from (13). Inasmuch as the polynomial $I(\varepsilon)$ is positive, its discriminant is $1 - 4 \langle s^2 \rangle \cdot \langle k^2 \rangle \le 0$ and this signifies that relation (12) holds true. Definition (4) suggests that, inasmuch as $\Delta E = a^2 p^2 / 2\rho$, it is necessary to assume $\langle k^2 \rangle = (1 + a^2) k^2$. Consequently, according to constraints (12),

$$\langle s^2 \rangle (1 + a^2) k^2 \geqslant \frac{1}{4}. \tag{14}$$

At the limit, fluctuation of a vortex becomes equal to its intrinsic dimension. Such a limit corresponds to the smallest possible, under constraint (14), vortex "radius"

$$\langle s^2 \rangle^{1/2} = \frac{1}{2} k^{-1} (1 + a^2)^{-1/2}. \tag{15}$$

A stationary structure of the fluctuation field is realized in the form of a standing wave with a phase difference π between neighboring nodes. Flow of the vortex medium with an appreciable momentum "content" $p = \rho U$ is realized within a narrow region of the wave packet, namely where superposition of elementary waves according to relation (7) results in their mutual amplification.

In the "boundary layer" approximation ($\partial/\partial y \gg \partial/\partial x$) Eqs. (3) and (5) become

$$\frac{\partial^2 a}{\partial y^2} + \left(\frac{\partial b}{\partial y}\right)^2 a^3 = 0, \quad \frac{\partial}{\partial y}\left(a^2 \frac{\partial b}{\partial y}\right) = 0. \tag{16}$$

In the absence of a characteristic dimension there exist in the mixing layer (Fig. 1) self-adjoint distributions of flow parameters [3] where the transverse dimension of the turbulence zone $\delta = \gamma x$, where γ is some constant. Equations (16) with $a = a(\varphi)$, $b = b(\varphi)$, and $\varphi = y/\delta$ yield

$$a^2 b' = c, \quad a a'' + c^2 = 0. \tag{17}$$

Here c is the integration constant. Since the change of φ over the entire layer width is equal to unity, the first of Eqs. (17) yields $\int a^2 db = c$. Taking into account relation (6), we obtain $c = 1$. The maximum of a occurs inside the layer, most likely within the region of contact with the stationary medium, where the probability of excitation of unstable perturbations is highest. The second of Eqs. (17) yields $a' = \pm[-2 \ln(a/a_0)]^{1/2}$, where a_0 is the maximum wave amplitude. We now introduce a variable r such that $a = a_0 \exp(-r^2)$. We then have

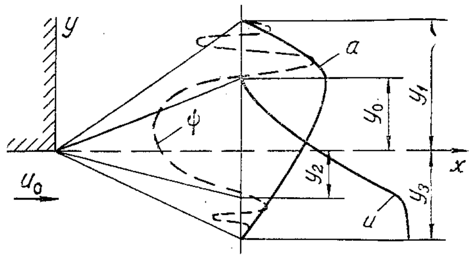

Fig. 1. Schematic diagram of mixing layer.

$$\frac{d\varphi}{dr} = \mp\sqrt{2}\,a_0 \exp(-r^2), \quad (18) \qquad \frac{db}{dr} = \mp\frac{\sqrt{2}}{a_0}\exp r^2. \quad (19)$$

The upper sign in expressions (18) and (19) corresponds to $\varphi<\varphi_0$, while the lower sign here correspond to the region $\varphi>\varphi_0$, with $\varphi=\varphi_0$ corresponding to the fluctuation peak (where $a' = 0$). Outside the mixing zone $a \to 0$ and $r \to \infty$. Normalizing the amplitude according to expression (6), with relation (19) taken into account, we obtain $a_0 = 1/\sqrt{2\pi} = 0.4$ as the result. The solution to Eq. (18) with $\varphi = 0$ at some $r = r_1$ can be written as

$$\varphi = \frac{1}{2}[\Phi(r_1)\mp\Phi(r)], \quad \left(\Phi(r) = \frac{2}{\sqrt{\pi}}\int_0^r \exp(-r^2)\,dr\right). \quad (20)$$

According to expression (19),

$$b = \mp\frac{\sqrt{2}}{a_0}D(r), \quad \left(D(r) = \int_0^r \exp r^2 dr\right). \quad (21)$$

At the point $\varphi = 0$ the wave number $|\partial b/\partial x| = \varphi/a^2x$ of the longitudinal wave becomes zero, which means that here the wave motion occurs in the transverse direction only and one can logically expect the mode of the transverse standing wave to change upon passage from the region $\varphi > 0$ to the region $\varphi < 0$. This is possible when two different half-waves coexist at the point $\varphi = 0$, i.e., one node of the ψ-wave is located here. The two half-waves adjoining this node correspond to intervals $0\leqslant\varphi\leqslant\varphi_0$ and $\varphi_2\leqslant\varphi\leqslant 0$, respectively, on each of which the phase changes by π. Therefore, according to expressions (21), $m\sqrt{\pi} = 2D(r_m)$ $(m = 1, 2, \ldots)$. Here $m = 1$ for the half-wave $0\leqslant\varphi\leqslant\varphi_0$; and $m = 2$ for the next half-wave $\varphi_2\leqslant\varphi\leqslant 0$. From tabulated values of $D(r)$ [4] we obtain $r_1 = 0.74$ and $r_2 = 1.1$; correspondingly $a = 0.23$ at $r = r_1$, $\varphi = 0$; and $a = 0.12$ at $r = r_2$, $\varphi=\varphi_2$. From solution (20) we determine the first node $\varphi_2 = -0.09$ of the wave in the region $\varphi < 0$. The maximum wave amplitude occurs at $r = 0$ and $\varphi_0 = 0.35$, correspondingly. We have $\varphi_3 = -0.15$ for the inner boundary and $\varphi_1 = 0.85$ for the outer boundary.

According to relation (15), the interval of vortex existence in a longitudinal wave with $k = |\partial b/\partial x|$ is $2\Delta x = |\partial b/\partial x|^{-1}(1 + a^2)^{-1/2}$. When $\partial b/\partial x = -\varphi/a^2x$, therefore, the longitudinal dimension (radius) of the vortex will be $\Delta x = a^2(1 + a^2)^{-1/2}x/2\varphi$. A freely expanding real vortex of the largest dimension must be able to occupy the entire width of the stream [3], within a certain bounded region where its motion is in a bound state. Such a vortex will have its maximum bounded dimension at the point $\varphi=\varphi_0$, so that $\Delta x = a_0^2(1 + a_0^2)^{-1/2}x/2\varphi_0$ (at $\varphi = 0$ this dimension Δx would be infinitely large and, therefore, we have assumed the maximum wave intensity to occur at a point $\varphi=\varphi_0 > 0$). Such a vortex is contained within the region $0\leqslant\varphi\leqslant\varphi_0$ and, therefore, $\Delta x = y_0 = \varphi_0\gamma x$. Combining the two definitions of Δx, we obtain $\gamma = a_0^2/2\varphi_0^2(1 + a_0^2)^{1/2}$ and, thus, $\gamma = 0.65$ as the result. Consequently, the fluctuation peak occurs at $y_0 = 0.212x$, the inner absolute boundary of fluctuations is located at $y_3 = -0.091x$, and the inner absolute boundary of the "throw-away" region in a large vortex is located at $y_2 = -0.0545x$. The outer absolute boundary of turbulence is located at $y_1 = 0.515x$. The absolute level of fluctuations drops sharply in regions $\varphi<\varphi_2$ and $\varphi>\varphi_0$. When the stream shifts within the region of large fluctuations $\varphi_2\leqslant\varphi\leqslant\varphi_0$, then the width of the mixing layer within the range of average flow is $y_0 - y_2 = (\varphi_0-\varphi_2)\gamma x = 0{,}27\,x$. This agrees with available experimental data [3, 5].

The average flow of the vortex medium will be made to obey the equations of motion and continuity

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \rho^{-1}\frac{\partial \tau}{\partial y}, \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (22)$$

with the longitudinal component u and the transverse component v of the velocity such that

$$u = \partial q/\partial y = u_0 f', \ v = -\partial q/\partial x = \gamma u_0 (\varphi f' - f), \ q = \gamma u_0 x f(\varphi),$$

where q is the flow function and u_0 is the velocity of the unperturbed stream. The shearing stress τ will be expressed as $\tau = \xi h \partial u/\partial y$, with the quantity ξh playing the role of eddy viscosity μ_T and ξ denoting the characteristic numerical parameter of the problem (explained in Appendix). In the "boundary layer" approximation $\rho u = h\partial b/\partial y$ and, thus, $\tau = \xi\rho(\partial b/\partial y)^{-1}u\partial u/\partial y$. According to Eq. (22),

$$ff'' + n^2(a^2 f' f'')' = 0, \quad n^2 = \frac{\xi}{\gamma}. \tag{23}$$

It is necessary to satisfy the conditions at the limits of the interval $\varphi_2 \leqslant \varphi \leqslant \varphi_0$, inasmuch as a wave packet with a sufficiently intense average motion must include not only the region of maximum fluctuation intensity but also the half-wave $\varphi_2 \leqslant \varphi \leqslant 0$, where momentum is retained in the main stream (otherwise the average motion would not spread over the region $\varphi > 0$).

At the boundary $\varphi = \varphi_2$, where v = 0 and u = u_0, we obtain

$$\varphi_2 = -0.09, \ f = \varphi_2, \ f' = 1. \tag{24}$$

At the lower boundary of average flow u = 0 and thus

$$\varphi = \varphi_0, \ f' = 0. \tag{25}$$

There also prevails the condition of smooth coupling between the given layer and the unperturbed stream

$$\varphi = \varphi_2, \ f'' = 0. \tag{26}$$

We now transform Eq. (23) to variables r, $\sigma(r) = f(\varphi)$. Then expression (18) yields

$$f' = \mp \frac{\exp r^2}{a_0 \sqrt{2}} \sigma', \ f'' = \frac{\exp 2r^2}{2a_0^2} (\sigma'' + 2r\sigma'), \tag{27}$$

and, consequently, we have

$$2(\sigma + n^2 r\sigma')(\sigma'' + 2r\sigma') + n^2[\sigma'(\sigma'' + 2r\sigma')]' = 0. \tag{28}$$

We will solve the problem (24)-(28) by the perturbation method. As the initial approximation we assume a linear velocity profile in the layer so that f" = $-\beta$ is some constant quantity. This initial approximation is, according to the second of relations (27), equivalent to the equation

$$\sigma'' + 2r\sigma' = 2\theta, \ \theta = -\beta a_0^2 \exp(-r^2) = \theta(r). \tag{29}$$

Inserting the expression (29) for 2θ instead of the corresponding expression in Eq. (28), we obtain the equation

$$\sigma'' - 2n\sigma' + \frac{2}{n^2}\sigma = 0, \tag{30}$$

a consequence of the approximation which ignores the curvature of the profile and, thus, a form of "compensation" of such perturbation of Eq. (28) through exaggeration of the curvature factor. The result of further refinement, viz. of averaging (adding) Eqs. (29) and (30)

$$\sigma'' + \frac{1}{n^2}\sigma = \theta(r), \tag{31}$$

is found to be acceptable. The constant β in the expression for $\theta(r)$ replaces the integration constant, which would have appeared here if the order of Eq. (28) could be reduced in a rigorous manner. The constant β can, therefore, be determined from one of the boundary conditions.

For the Eq. (31) we will now construct the inner limiting solution $\sigma = \sigma_2$ at $r \to r_2$ and the outer limiting solution $\sigma = \sigma_0$ at $r \to 0$

$$\sigma_0 = c_1 \cos(r/n) + c_2 \sin(r/n) + n^2\theta(0),$$
$$\sigma_2 = c_3 \cos(r/n) + c_4 \sin(r/n) + n^2\theta(r_2).$$

The arbitrary constants c_1, ..., c_4 must be such that the limiting solutions will coincide at the boundaries of the $(0, r_2)$ interval. Taking this into consideration, we obtain the composite solution, put in the form of the sum of both limiting ones, corrected by subtraction of

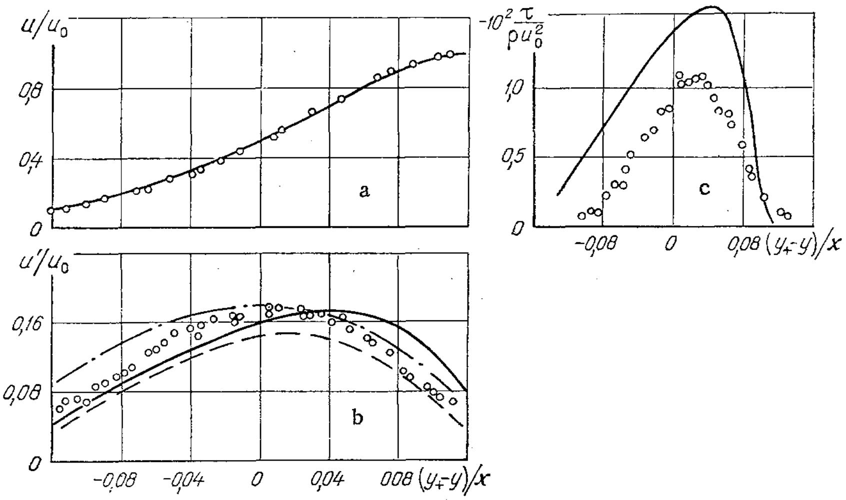

Fig. 2. Profiles of (a) longitudinal velocity, (b) velocity fluctuations, and (c) shearing stress.

the inner limit of the outer solution $\sigma_0(r_2)$ [6]:

$$\sigma = \frac{0.17n}{\sin(r_2/n)}\cos(r/n) - \frac{0.17n\cos(r_2/n)}{2\sin(r_2/n)} + \frac{n^2}{2}[\theta(0) + \theta(r_2)]. \qquad (32)$$

Here $\theta(0) + \theta(r_2) = -0.17\beta$. Equation (32) takes into account the condition $\sigma' = 0$ at the outer boundary $r = 0$ of the layer, this condition following from conditions (25), as well as the condition $\sigma' = -a_0\sqrt{2}\exp(-r_2^2) = -0.17$ at its inner boundary $r = r_2$ based on the second of conditions (24). It follows from conditions (26) that at $r = r_2 = 1.1$ we have $\sigma'' = -2r\sigma'$ Therefore, one can write the equation $2nr_2 \operatorname{tg}(r_2/n) = -1$, which yields $n = 0.464$ and then $\xi = \gamma n^2 = 0.13$, as well. The first of conditions (24) signifies that $\sigma = -0.09$ at $r = r_2$ and thus, according to Eq. (32), $\beta = 5.9n^{-2}[0.18 + 0.17n \operatorname{ctg}(1.1/n)]$. Calculations yield $\beta = 2.72$ on this basis. We finally have

$$f(\varphi) = \sigma(r) = 0.113\cos(2.15r) - 0.009, \qquad (33)$$

$$\frac{u}{u_0} = f'(\varphi) = -\frac{\exp r^2}{a_0\sqrt{2}}\sigma'(r) = 0.43\exp r^2\sin(2.15r). \qquad (34)$$

According to expression (33), the zero flow line corresponds to $r = 0.693$ or $\varphi = 0.015$, $y/x = 0.0091$ here.

The calculated profile of average velocity is shown in Fig. 2a, where dots represent experimental data [5]. The values of y_+ correspond to $u = 0.5u_0$; $y_+ = 0.051x$ according to calculation, and $y_+ = 0.054x$ according to the experiment [7].

Inasmuch as $\rho a^2U^2/2$ is the mathematical expectation of fluctuation energy by definition, $a^2U^2 \simeq a^2u^2$ will be the dispersion of deviations of the forward velocity, and the relative rms intensity of fluctuations of the forward velocity will thus be $u'/u = a$. The theoretical profile of u'/u_0 is compared in Fig. 2b with the experimental data according to R. P. Patel [5], along with the experimental Lipman relation (dash line) and Wygnanski relation (dash-dot line) [5, 7]. The theoretical relation $\tau/\rho u_0^2 = \xi a^2 f'f''$ or

$$-\frac{\tau}{\rho u_0^2} = 0.1\xi\exp r^2\sin(2.15r)[r\sin(2.15r) + 1.075\cos(2.15r)]$$

is also compared with experimental data [5] (dots in Fig. 2c).

APPENDIX

Just as in the kinetic theory, we have $\mu_T \sim \rho u' l$ with u' denoting the fluctuation of velocity on the interval of $l \sim (\partial b/\partial u)^{-1}$ of vortex-particle oscillation (both quantities are statistical means). Spontaneous appearance of large vortices is caused by action of such velocity fluctuation in the direction tangent to a closed contour, which thus becomes the contour of the vortex [3]. For a circular contour with radius $L \sim \Delta x$, for instance, the likely (in the rms sense) circulation Γ will be such [3] that

$$u' = \frac{\Gamma}{2\sqrt{2\pi l L}},\ \Gamma = \langle (\oint w dl)^2 \rangle^{1/2},\ u' \equiv \langle w^2 \rangle^{1/2}.$$

Here w is the velocity fluctuation on a contour element dl. It is legitimate to propose that the measure of rotational excitation in a vortex is determined by parameter h, similar to the Planck constant in quantum physics — an analog of our h — serving as the measure of the inner angular momentum of a rotating particle. We then have $h \sim \rho\Gamma$ (in a nonvortical stream $h \to 0$ as $\Gamma \to 0$). On the basis of this estimate we obtain $\mu_T = \xi h$ and $\xi \sim 0.5(l/\pi L)^{1/2}$. For instance, $\xi = 0.1$ when $L = 5l$. The circulation Γ is expressed here in the "isotropic turbulence" approximation and, therefore, the preceding treatment is appropriate when no strong inhomogeneities occur in the fluctuations as, for example, in free turbulent streams.

NOTATION

Here ψ is the wave function; a, wave amplitude; b, wave phase; U, modulus of the velocity; ρ, density (incompressible fluid); h, "quantum" parameter; x, y, longitudinal coordinate and the transverse coordinate in the mixing layer; and t, time.

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FLOW MODEL OF A BOILING LIQUID IN NOZZLES

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A nonequilibrium mathematical model of flow is constructed for a boiling liquid in nozzles. The theoretical results are compared with empirical results for the case of flow of boiling water.

Present theoretical models of flow of a boiling liquid in nozzles (tubes) mainly aim to determine the maximum flow rate. A critical survey of these models is contained in [1, 2]. It is seen that homogeneous equilibrium and metastable models are the most widely used, along with a model allowing for discrete phase flow with relative slip and simple models considering the thermodynamic nonequilibrium of the process using empirical coefficients. It was shown in [3] that the use of such models is limited by the complexity of the flow structure of a boiling liquid and the nonequilibrium of exchange processes between the phases. In connection with this, it is important that a theoretical model of boiling liquid flow in nozzles be constructed which considers the structure of the flow and the effects of nonequilibrium of the interphase transfer. Such a model should describe the origination of the vapor phase in the liquid flow and the combined flow of the vapor and liquid phases.

Well-known experimental investigations of the structure of a boiling liquid flow in Laval nozzles [4-8] with moderate initial parameters show that boiling begins primarily on the nozzle walls. The method in [9] is used to determine the intensity of this vapor formation. It is assumed that vapor bubbles are generated until the vapor content of the mixture reaches a value which is limiting for the existence of a bubble structure ($\alpha \simeq 0.74$).

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